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Floquet Reference Solutions for the Lunar Theory and the Jovian Moons

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Reference solutions are constructed for lunar theory and Jupiter's Galilean satellites, consisting of a periodic orbit and associated Floquet modes. Although these solutions are obtained numerically, they introduce a full set of literal variables for perturbation work. In the Jovian problem, difficulties with resonances are avoided, since perturbations are expanded in the Poincaré exponents, not the Keplerian frequencies. The modes for the inner three Jovian satellites are discussed in detail.

Introduction

WITH the notable exception of Hill's[1] lunar theory, virtually all work in celestial mechanics is based on two body reference orbits, either explicitly or implicitly. The explicit use of two body orbits is the foundation of most of the classic works in the field. The implicit use of such orbits arises, for example, when a periodic orbit is transformed to Keplerian variables in order to use standard developments of the perturbative function. In this paper we examine the use of the complete periodic orbit/Floquet mode reference solution as an alternative to the use of Keplerian orbits. We discuss the lunar theory and the problem of Jupiter's satellites, and compare this technique to the method of secular perturbations.

Dynamics

We have elected to employ Jacobi coordinates for both problems. In these coordinates, the motion of body j is referred to the center of mass of the first $j-1$ objects. Thus, if $\boldsymbol{r}_i$ is the inertial position vector of body i, then Jacobi position vectors are defined by

$$\boldsymbol{\rho}_{j-1} = \boldsymbol{r}_j - \frac{1}{M_{j-1}} \sum_{i=1}^{j-1} m_i \boldsymbol{r}_i \tag{1}$$

where

$$M_i = \sum_{j=1}^{i} m_j \tag{2}$$

These relations may be inverted to yield the inertial position vectors as functions of $\boldsymbol{\rho}_i$. Then the kinetic energy of the N-body system has the form

$$T = \frac{1}{2} \sum_{i=1}^{N-1} \beta_i \dot{\boldsymbol{\rho}}_i^2 \tag{3}$$

where

$$\beta_i = M_i m_{i+1} / M_{i+1} \tag{4}$$

The system Hamiltonian then takes the form

$$H = \sum_{i=1}^{N-1} \frac{\boldsymbol{P}_i^2}{2\beta_i} + V \tag{5}$$

where V is the system potential and

$$\boldsymbol{P}_i = \beta_i \dot{\boldsymbol{\rho}}_i$$

The interparticle position vectors $\boldsymbol{r}_{ij}$ are given by

$$\boldsymbol{r}_{ij} = \frac{M_{j-1}}{M_j} \boldsymbol{\rho}_{j-1} - \frac{M_{i-1}}{M_i} \boldsymbol{\rho}_{i-1} + \sum_{k=i}^{j-1} \frac{m_{k+1}}{M_{k+1}} \boldsymbol{\rho}_k \tag{6}$$

where $j > i$ and $M_0 = 0$ when it occurs in Eq. (6).

For the main problem of the lunar theory, the three bodies considered are the Earth, moon, and sun, respectively. By a well-known argument, Brouwer and Clemence,[2] the motion of the sun is modeled to high accuracy as a two-body elliptical orbit. It is not necessary to follow Hill[1] in placing the sun at infinite distance in order to obtain a well-posed periodic orbit problem. It is, however, necessary to exclude solar eccentricity effects since these would introduce a second (one-year) frequency into the problem, and destroy the existence of the desired reference orbit with a period of one synodic month.

Therefore, using a coordinate system rotating at the solar mean motion about the pole of the ecliptic, and expanding in powers of the solar eccentricity, the main problem of the lunar theory has the Hamiltonian

$$K = K_0 + e_\odot K_1 + \frac{1}{2!} e_\odot^2 K_2 + \cdots \tag{7}$$

The desired periodic reference orbit will be a solution of

$$\begin{aligned} K_0 = {} & (1/2\beta_1) \boldsymbol{P}_1^2 + n_\odot (P_{1x}\rho_{1y} - P_{1y}\rho_{1x}) \\ & - G m_1 m_2 |\boldsymbol{\rho}_1|^{-1} - G m_1 m_3 |\boldsymbol{\rho}_2 + (m_2/M_2)\boldsymbol{\rho}_1|^{-1} \\ & - G m_2 m_3 |\boldsymbol{\rho}_2 - (m_1/M_2)\boldsymbol{\rho}_1|^{-1} \end{aligned} \tag{8}$$

where

$$\boldsymbol{\rho}_2^T = (a_\odot, 0, 0)$$

The Floquet reference solution for this problem will incorporate terms in K_0 through the second order, relegating solar eccentricity terms (K_1, K_2, etc.) and third- and higher-order terms in K_0 to the perturbation solution.

Problem formulation for the Galilean satellites of Jupiter is dominated by the fact that the inner three moons are involved in an apparently exact resonance. The fourth moon, Callisto, is not involved in this resonance. (Lieske[3] has reported a 3:7 commensurability between Ganymede and Callisto, but it does not involve an actual libration. Problem formulation

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would actually be simpler if it did, since all four moons could then be treated at once.) Thus, the reference solutions for the Jovian moons will involve two separate periodic orbits. An "inner" problem will involve Jupiter and its inner three moons, and an "outer" problem will treat Callisto.

It is highly desirable to include all possible effects into the reference solution. To this end, the inner problem can include the effects of the known spherical harmonic coefficients of the jovipotential. The outer problem cannot include the inner three moons without destroying periodicity, but it can be formulated as a lunar-theory type of problem, with the jovipotential included as well.

For the inner problem, we introduce a frame rotating clockwise at the apsidal rate Ω (positive) of the periodic orbit. Then,

$$K_1 = \sum_{i=1}^{3}\left(\frac{P_i^2}{2\beta_i} + \Omega(P_{iy}\rho_{ix} - P_{ix}\rho_{iy})\right) - \sum_{i=1}^{4}\sum_{j>i}^{4}\frac{Gm_i m_j}{|r_{ij}|} + \sum_{i=2}^{4}\sum_{l=2} m_1 m_i J_l\left(\frac{R_J l}{r_{1i} l+1}\right)P_l(\sin\phi_i) \tag{9}$$

Unlike the lunar problem, this Hamiltonian possesses two integrals of the motion K_1, and the Z component of angular momentum. Therefore, it requires the specification of two parameters, Ω and the period τ_1, in order to choose a particular periodic orbit.

In order to extract a lunar-type model for the Callisto problem, the Jupiter-Callisto radius vector $\boldsymbol{r}_{15}$ and the Jupiter-sun vector $\boldsymbol{r}_{16}$ are expanded in powers of the $\boldsymbol{\rho}$ vectors of the inner three moons, and these center-of-mass terms are relegated to the perturbation solution. The correct β and mass parameters are used whenever possible, leading to the Hamiltonian for the outer problem

$$K_0 = P_4^2/2\beta_4 + n_\odot(P_{4x}\rho_{4y} - P_{4y}\rho_{4x}) - \frac{Gm_1 m_5}{|\rho_4|} - \frac{Gm_6 m_5}{|\rho_5 - (M_4/M_5)\rho_4|} - \frac{Gm_1 m_6}{|\rho_5 + (m_5/M_5)\rho_4|} + m_1 m_5 \sum_l J_l\left(\frac{R_J l}{\rho_4 l+1}\right)P_l(\sin\phi_4) \tag{10}$$

The solar position vector $\boldsymbol{\rho}_5$ takes the same form as in the lunar theory when restricted to zero eccentricity and inclination.

Introduction of Modal Variables

In an earlier work,[4] we gave the theory behind the use of Floquet modes in a perturbation calculation. The canonical variables are assembled into a state vector X, where for definiteness we assume conjugate q_i, p_i are paired. Having constructed a periodic orbit X_p, the Hamiltonian may be expanded about this orbit to give

$$H(\delta x,t) = (1/2!)H_{uv}\delta x_u \delta x_v + (1/3!)H_{uv\lambda}\delta x_u \delta x_v \delta x_\lambda + \cdots \tag{11}$$

where δx_u is a component of $\boldsymbol{\delta x}$, the displacement from the periodic orbit, and the summation convention is used.

The Floquet problem is generated by restricting the Hamiltonian to its quadratic terms. The solution to this problem may be written

$$\boldsymbol{\delta x}(t) = \Lambda(t)e^{Jt}\boldsymbol{a} \tag{12}$$

The matrix Λ is periodic and will consist of periodic vectors Λ_i, the modal eigenvectors. The vector $\boldsymbol{a}$ consists of canonically conjugate pairs of variables, which are actually constant in the absence of perturbations. The constant matrix J is the Jordan normal form of the monodromy matrix $\Phi(\tau,0)$, where Φ satisfies the differential equation

$$\dot{\Phi} = ZH_{uv}\Phi \qquad \Phi(0,0) = I \tag{13}$$

The matrix Z is block diagonal, with diagonal blocks of the form

$$\begin{Bmatrix} 0 & 1 \\ -1 & 0 \end{Bmatrix}$$

and all off-diagonal blocks zero.

For use in perturbation theory, it is necessary to construct the periodic orbit $X_p(t)$, the Poincaré exponents (diagonal elements of J), and the periodic matrix $\Lambda(t)$. Terms of third order and higher in the Hamiltonian must then be transformed into the new modal variables, and neglected effects must be included before perturbation work can begin.

The complex-valued elements of the vector $\boldsymbol{a}$ are local constants of the motion, and are canonical in pairs, a_{2i-1}, a_{2i}. Thus, they may serve as a local set of elements for the particular periodic orbit used in the development of the problem. However, they carry both amplitude and phase information for each mode, and it may be convenient to display this explicitly by transforming to real-valued, action-angle variables. Thus, if W_i is the absolute value of a purely imaginary Poincaré exponent, then such an oscillatory mode can be put in the form

$$\sqrt{2P_i/W_i}\{\Lambda_r(t)\sin Q_i + \Lambda_I(t)\cos Q_i\} \tag{14}$$

where $\Lambda_r(t)$ and $\Lambda_I(t)$ are the real and imaginary parts of the appropriate column of $\Lambda(t)$, and the new canonical variables Q_i, P_i have equations of motion

$$\dot{P}_i = 0 \qquad \dot{Q}_i = W_i \tag{15}$$

in the absence of perturbations.

The choice of action-angle variables P_i, Q_i vs rectangular variables a_{2i-1}, a_{2i} is similar to the choice between e, w, vs h, k in the classical two-body problem. If P_i can go to zero, $\dot{Q}_i$ can contain infinite terms due to perturbations. On the other hand, if P_i is moderately large, secular terms are easily incorporated in a Q-perturbation expression, while they are unacceptable in a_i coordinates.

There will be two zero Poincaré exponents for every exact constant of the motion in the original problem. Such a degenerate mode has the representation

$$\Lambda_j(t)a_j + \{\Lambda_{j+1}(t) + (t-t_0)\Lambda_j(t)\}a_{j+1} \tag{16}$$

with a secular term. The presence of such terms does not present a practical difficulty, however. In the lunar theory, for example, the zero exponents are associated with conservation of K_0. The first term in Eq. (16) then locally represents a small change in epoch time, which is easily handled globally by replacing t with $t-t_0$ in all Fourier series. The second (secular) term results if the "wrong" periodic orbit is used for the problem. Since this is essentially a choice of synodic period of the moon, an accurately known quantity, we expect that $a_{j+1} \approx 0$. This secular term can also be eliminated if the Fourier coefficients of the periodic orbit are developed in powers of $K - K_0$, with a similar development for the frequency of the periodic orbit. We then arrive at a familiar situation in classical celestial mechanics where the semimajor axis is the variable normally found both in the coefficients and arguments of trigonometric terms. Here, $K - K_0$ fills this role. The standard technique used in that instance will work here also, and it is never necessary to resort to "intrinsic" forms to eliminate the secular terms. For the inner three Jovian moons, a second degenerate mode arises from the conservation of angular momentum. This mode can also be eliminated by the same method.

Numerical Considerations

In any undertaking that depends heavily on numerical work, it is absolutely necessary to incorporate as many checks on correctness and accuracy as possible. This, in itself, is sufficient justification for using Hamiltonian dynamics, since the symmetries inherent in this approach furnish many useful controls. To begin, each $H_{uv}\cdots$ is a completely symmetric tensor, which reduces the labor needed to derive it. Each order tensor is also the partial derivative of the previous order tensor, which means that its coding can be efficiently validated by numerically differentiating the previous order tensor. Hand checking is thus reduced to verifying that the Hamiltonian function is correctly programmed. Since the inner three moons of Jupiter require the integration of 342 scalar differential equations to calculate the full-state transition matrix Φ, making full use of any symmetries is necessary.

The techniques of constructing a periodic orbit have been discussed by many authors and will not be treated here. Two points unique to the ephemeris application deserve mention, however. First, absolute accuracy is a function of convergence of the periodic orbit algorithm and of step size used in the numerical integration. Small residuals and corrections to initial conditions in no way guarantee that the orbit is very accurate, they only demonstrate that the final conditions are well-behaved functions of the initial conditions. Second, the desired end products of the entire calculation are the periodic functions X_p and Λ_i, and the Poincaré exponents W_i. Since W_i are frequencies, their accuracy sets the period of long-term validity of the theory. This virtually requires that the periodic orbit be constructed in double precision.

One valuable check on the accuracy of the periodic orbit X_p is furnished by the harmonic analysis used to convert the numerical integration to a form suitable for ephemeris work. Brown and Shook[5] give the error analysis for this algorithm. However, when the sample points are corrupted by numerical noise, the algorithm attempts to fit the noise with high-order harmonics, with the result that the amplitude of high-order harmonics indicate, in a general way, the accuracy of the integration.

The Floquet calculation begins with the computation of the state transition matrix Φ. The correctness of the Φ integration can be checked by numerically calculating the partial derivatives of a short arc of trajectory, since

$$\Phi(t_2,t_1) = \left.\frac{\partial X(t_2)}{\partial X(t_1)}\right|_{X_p} \tag{17}$$

However, a much more sensitive accuracy control is furnished by the fact that the state transition matrix for a Hamiltonian system is symplectic, that is

$$\Phi^T Z \Phi = Z \tag{18}$$

This relation also furnishes an inversion formula for Φ, which can be used to reduce the required span of numerical integration. If the orbit is symmetric, then

$$\Phi(t,0) = A\Phi(-t,0)A \tag{19}$$

where A is a matrix with ± 1 on its diagonal, depending on whether each coordinate is an even or odd function. These results then enable us to express the monodromy matrix as

$$\Phi(\tau,0) = -ZA\Phi^T(\tau/2,0)AZ\Phi(\tau/2,0) \tag{20}$$

which requires only half the integration interval, and does not require a matrix inversion. Further computational efficiency can be obtained since the vertical motion is decoupled from the planar motion at this order.

As is well known, the Poincaré exponents W_i are related to the eigenvalues α_i of $\Phi(\tau,0)$ by

$$W_i = |\ln(\alpha_i)/\tau| \tag{21}$$

Again, several results on Hamiltonian systems provide accuracy checks. There are two zero W_i for each constant of the motion in the original system, although this check must be applied with care since most eigenvalue routines experience some difficulties with multiple eigenvalues. Of far more value are the repeated eigenvectors associated with zero W_i, since Hamiltonian theory predicts their values. For example, the repeated eigenvector associated with conservation of H itself is just $\dot{X}_p$ $(t=0)$.

The final step is the calculation of the modal vectors $\Lambda_r(t)$ and $\Lambda_I(t)$ for each mode. Since expression (14) also satisfies the differential equation (13), it is easy to show that the real and imaginary vectors satisfy

$$\begin{aligned} \dot{\Lambda}_r(t) &= ZH_{uv}\Lambda_r(t) + W_i\Lambda_I(t) \\ \dot{\Lambda}_I(t) &= ZH_{uv}\Lambda_I(t) - W_i\Lambda_r(t) \end{aligned} \tag{22}$$

Initial conditions at $t=0$ are supplied by the complex-valued eigenvectors output in the calculation of the Poincaré exponents. Since Eqs. (22) are coupled, the vectors $\Lambda_r(t=0)$ and $\Lambda_I(t=0)$ for each mode must be normalized by a common factor. We have chosen to set

$$|\Lambda_r(t=0)|^2 + |\Lambda_I(t=0)|^2 = 1 \tag{23}$$

although this may hide the physical significance of the mode in those cases where it can be interpreted as an eccentricity or inclination. A final accuracy control is provided by the fact that the Λ are periodic functions, returning to their initial values in one period. Results of this numerical integration are then converted to Fourier series via harmonic analysis.

Modal eigenvectors for modes with zero W_i are best constructed from their predicted functional form. As was shown in Ref. 4, if the Hamiltonian itself is the only constant of the motion, then the repeated eigenvector is

$$\Lambda_i(t) = \dot{X}_p \tag{24}$$

while the associated generalized eigenvector is

$$\Lambda_{i+1}(t) = -\tau^2 \frac{\partial X_p(t)}{\partial \tau} \tag{25}$$

The first is easily obtained by differentiating the Fourier series for X_p, while the second can be constructed by numerical differentiation if orbits for slightly different τ values are constructed. These modal variables a_i then have definite physical interpretations and the Λ vectors are not normalized.

Results

Expression (14) can be used to visualize each mode and thereby gain some insight into its physical significance. Since the modal frequencies W_i are much smaller than the orbital frequency $2\pi/\tau$, it is possible to draw plots of the periodic orbit combined with one mode, for constant values of the modal phase Q_i. These diagrams, combined with examination of the Fourier series coefficients for the modal vectors Λ, establish the physical characteristics of the modes.

For the lunar theory, the planar mode is basically a Keplerian eccentricity, while the vertical mode is an inclination. This can be established from the Λ Fourier series alone, which are dominated by terms that can be interpreted as the rectangular coordinate series coefficients for an inclined elliptical orbit, truncated to first order in e and i. This, in itself, furnishes a check on the calculations, but this is not

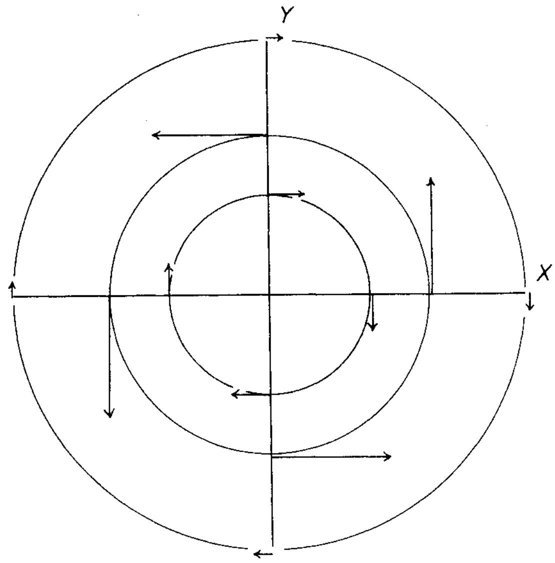

Fig. 1 Lowest frequency mode is the "great libration," a set of in-track oscillations.

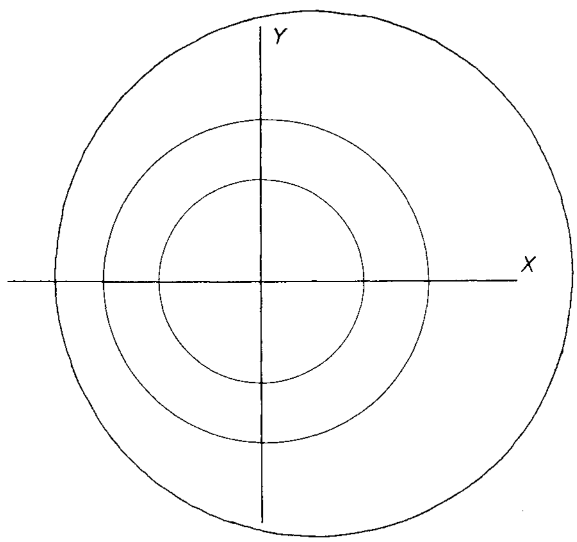

Fig. 2 Second planar mode is basically an eccentricity of Ganymede, with the line of apsides precessing every 1.321 yr.

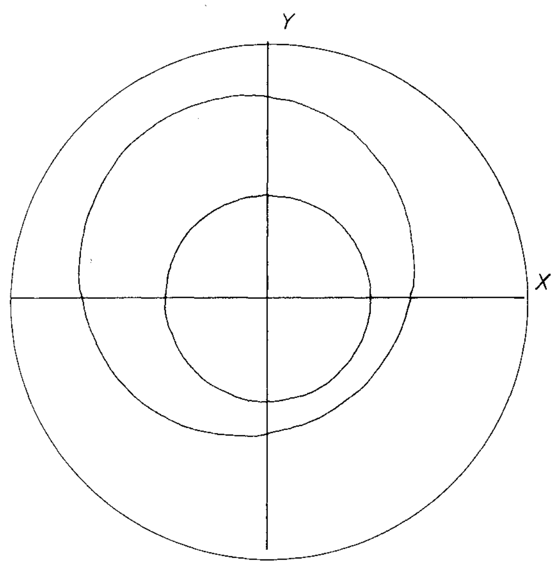

Fig. 3 Third planar mode combines eccentricities of Europa and Io with perijoves opposed.

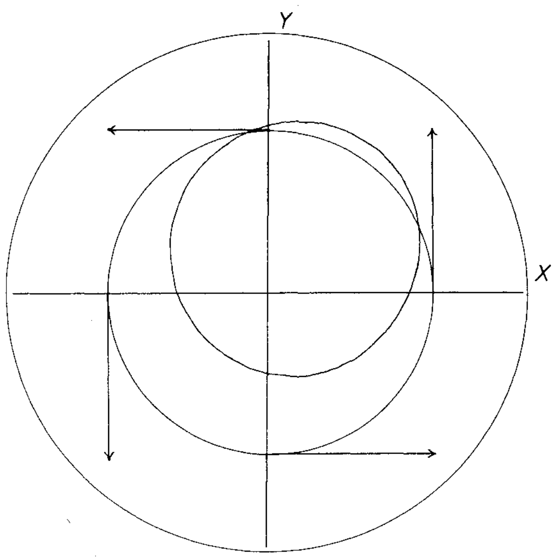

Fig. 4 An eccentricity of Io is combined with a much larger in-track oscillation of Europa in the final planar mode.

the complete interpretation of these modes. The modal frequencies, when referenced to inertial space, imply modal periods of 8.725 yr for the planar mode, and 18.703 yr for the vertical mode. These are the familiar periods for the advance of the perigee and regression of the node in the lunar theory. In both modes the Λ_r and Λ_l functions are not simple even-odd compliments of each other. For example, the inclination varies by over 7% depending on the phase of the vertical mode. Also, there are significant contributions to the modeshape functions Λ_i at frequencies other than those expected from Keplerian expansions.

In the Jovian moon problem the situation is markedly different, in that some modes exhibit non-Keplerian collective effects. These modeshapes are shown in Figs. 1-7 for the inner problem, superimposed on the periodic orbit. The mode is drawn as a smooth curve or as a set of displacement vectors, as appropriate.

Figure 1 shows the lowest-frequency planar mode, which is the "great libration" of the mean longitudes in the combination

$$-\lambda_{III}+3\lambda_{II}-2\lambda_I$$

This mode has never been unambiguously observed in the actual system because its amplitude appears to be very small. Its description does arise quite naturally from the Floquet analysis. The period of this mode is 5.648 yr, in agreement with classical analysis.

Figures 2-4 show the remaining three planar modes for the inner Jovian moons. The mode in Fig. 2 is basically a free eccentricity of Ganymede, with the line of apsides precessing every 1.321 yr in the rotating XY frame. The third planar mode (Fig. 3) combines eccentricities of Europa and Io with perijoves opposed. The period of this mode is 1.266 yr. The final planar mode is a significant departure from classical secular perturbation results. As shown in Fig. 4, it combines an eccentricity of Io with a much larger in-track oscillation of Europa. (The apparent intersection of the orbits is an artifact of the large amplitude used to plot the mode.) The period is 1.105 yr.

Figures 5-7 display the three out-of-plane modes, with periods of 1.34, 1.39, and 1.62 yr, respectively. Each mode represents a free inclination of one moon combined with the forced response of the other moons. The Poincaré exponent, in each case, gives the regression rate of the coincident lines of nodes.

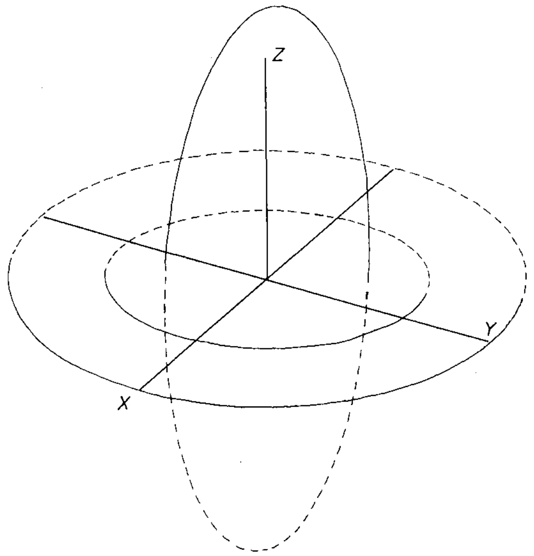

Fig. 5 Lowest-frequency vertical mode is an inclination of Io combined with smaller inclinations of Europa and Ganymede.

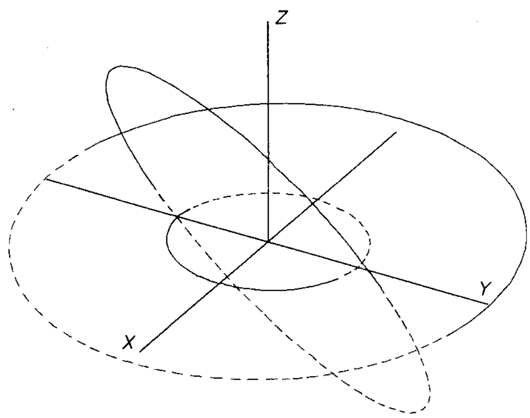

Fig. 6 Second vertical mode is dominated by the inclination of Europa.

Summary and Conclusions

It is possible to obtain, by a series of numerical calculations, the complete solution to the variational equations in the vicinity of a periodic orbit. Although numerical in character, these calculations introduce a complete set of canonical variables (modal variables) in literal form. The periodic orbit X_p and modal vectors Λ are periodic in time, so the numerical errors introduced in their calculation and reduction to Fourier series are bounded and periodic. The method also introduces a full compliment of frequencies for the problem, so the new Hamiltonian is completely nondegenerate. For the lunar theory, the three frequencies are the basic orbit frequency $2\pi/\tau$, and the two Poincaré exponents associated with the motion of the lines of modes and apsides. For the inner three Jovian moons there are seven nonzero Poincaré exponents associated with modes of the system. The required total of nine frequencies is found by including the orbital frequency $2\pi/\tau$ and the apsidial rate Ω of the periodic orbit. Since all nine frequencies are incommensurable, no small divisor problems should appear in the perturbation analysis.

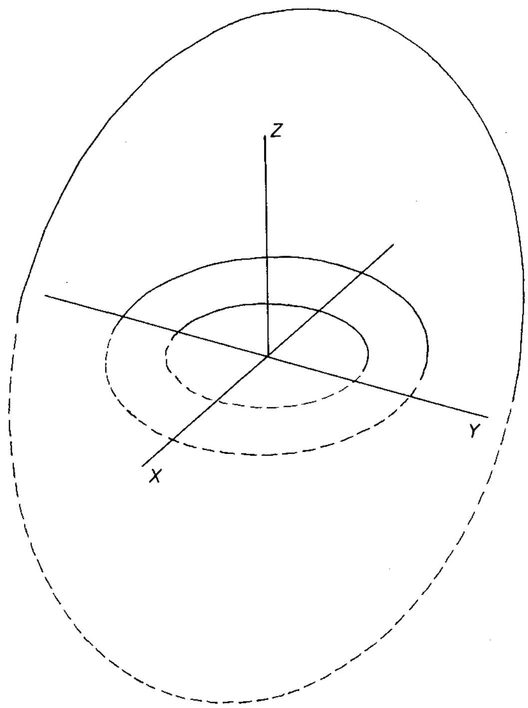

Fig. 7 Final vertical mode is the inclination of Ganymede.

The closest analogy with the method of Floquet modes is the method of secular perturbations. To compare these methods, we first note that the reference solution in secular perturbations is a set of uncoupled elliptic orbits, while the current method uses a periodic orbit. Both methods solve a set of linear differential equations via an eigenvalue/eigenvector problem, but the current method introduces one mode for each degree of freedom in the problem, while secular perturbations deal only with e, ω, i, and Ω, leaving the semimajor axis a and mean longitude λ unaltered. Secular perturbations are thus incapable of predicting the great libration, which is a purely in-track oscillation. Also, secular perturbations select only a few "secular" terms from the disturbing function for analysis, while the method of Floquet modes eliminates all terms in the system Hamiltonian through the second order. Finally, further perturbation work is normally done in some variant of the classical elements if secular perturbations are used. The Floquet analysis, on the other hand, introduces a complete set of canonical variables suitable for further perturbation work, and uniquely adapted to the problem under study.

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